Affine and fundamental vector fields

Bozhidar Z. Iliev [*] [†] [‡]

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[*]Laboratory of Mathematical Modeling in Physics, Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Boul. Tzarigradsko chaussée 72, 1784 Sofia, Bulgaria
[†]E-mail address: bozho@inrne.bas.bg
[‡]URL: http://theo.inrne.bas.bg/∼bozho/

Contents

Abstract

This is a review with examples concerning the concepts of affine (in particular, constant and linear) vector fields and fundamental vector fields on a manifold. The affine, linear and constant vector fields on a manifold are shown to be in a bijective correspondence with the fundamental vector fields on it of respectively general affine, general linear and translation groups (locally) represented on the manifold via the described in this work left actions; in a case of the manifold $\mathbb{K}^n = \mathbb{R}^n, \mathbb{C}^n$, the actions mentioned have the usual meaning of affine, linear and translation transformations.

1. Introduction

This is a review with examples concerning the concepts of affine (in particular, constant and linear) vector fields and fundamental vector fields on a manifold.

The linear vector fields are investigated in some detail in section 2. On this ground, the constant and, more generally, affine vector fields are briefly studied in section 3. Section 4 is devoted to the flows of vector fields. As an example, the flows of affine fields are found. Section 5 is devoted to the canonical parameters and invariants. As a major example, these functions for a constant vector field are found, which can serve a guiding line for finding them for arbitrary affine fields. Section 6 deals with fundamental vector fields in the context of affine vector fields, which turn to be such for particular representation of the affine group of $\mathbb{R}^n$. All of the results in the paper are illustrated (or introduced at first) for the case of the manifold $\mathbb{K}^n = \mathbb{R}^n, \mathbb{C}^n$.

In this paper, the following notation will be used. By $\mathbb{K}$ is denoted the field of real or complex numbers, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. A C^1 $\mathbb{K}$-manifold will be denoted by M, (U, u) is a local chart of M and $\{u^i\}$ is the coordinate system corresponding to it. The Latin indices $i, j, k \dots$ run from 1 to $\dim M$ and the Einstein's summation rule is assumed. The space tangent to M at $x \in M$ is $T_x(M)$. The set of vector fields on M is $\mathfrak{X}(M)$. If f is a C^1 mapping between C^1 manifolds, the induced by it tangent mapping is $T(f) := f_*$ and its restriction to $T_x(M)$ is $T_x(f) := f_*|_x$. An arbitrary path with domain J, $J \subseteq \mathbb{R}$ being an arbitrary real interval, is denoted by $\gamma \colon J \to M$.

2. Linear vector fields

The importance of the linear vector fields comes from the facts that a nonvanishing smooth vector field is locally linear and that their flows are governed by simple equations (in suitable coordinates) which can be solved explicitly. Usually, the linear vector fields are defined and considered in $\mathbb{R}^n$ equipped with standard Cartesian coordinates, as, e.g., in [1, p. 29], but below we are going to consider them on arbitrary C^1 manifolds.

Definition 2.1. Let X be a vector field on a C^1 manifold M, (U, u) be a local chart of M and $\{u^i\}$ be the local coordinates system associated with it. A vector field $X \in \mathfrak{X}(M)$ with local representation

$$X = X^i \frac{\partial}{\partial u^i} \tag{2.1}$$

is called *linear relative to* (U, u) *or to* $\{u^i\}$ if its local components $X^i = X(u^i)$ relative to $\{u^i\}$ have a $\mathbb{K}$-linear dependence on the coordinate functions $u^1, \dots, u^{\dim M}$, viz.

$$X^i = C^i_j u^j = \sum_{j=1}^{\dim M} C^i_j u^j \tag{2.2}$$

for some numbers $C^j_i \in \mathbb{K}$ forming the constant matrix $C := [C^j_i]$.

The existence of linear (local) vector fields is evident: if we fix $\{u^i\}$, then (2.1) and (2.2) define for a constant matrix C a vector field on U which is linear relative to $\{u^i\}$. The interesting problem is: which vector fields admit chart(s) with respect to which they are linear?

Proposition 2.1. *Let X be a C^1 vector field on a C^1 manifold M, $p \in M$, and $X_p \neq 0$. Then there exists a chart (U, u) such that $U \ni p$ and X is linear relative to (U, u).*

Proof. According to [2, proposition 1.53] or [1, p. 30], there is a chart (V, v) of M such that $V \ni p$ and

$$X|_V = \frac{\partial}{\partial v^1}. \tag{2.3}$$

Consider a chart (U, u) of the manifold M with $U \subseteq V$ and coordinate functions $u^i = f^i(v^2, \dots, v^{\dim M})e^{a^i v^1}$ (do not sum over i!) for some suitable C^1 functions f^i, where $a^i \in \mathbb{K}$ are constants and $a^1 \neq 0$. [1] According to (2.3), we have

$$X|_U = \frac{\partial}{\partial v^1}\Big|_U = \frac{\partial u^i}{\partial v^1}\Big|_U \frac{\partial}{\partial u^i}\Big|_U = \sum_i a^i u^i \frac{\partial}{\partial u^i}.$$

Therefore X is linear relative to the chart (U, u) with local coordinates $\{u^i\}$. In the particular case, the matrix C is the diagonal matrix $\operatorname{diag}(a^1, \dots, a^{\dim M})$. □

Corollary 2.1. *If X is a regular vector field on M, in a neighborhood of each point in M there is a chart relative to which X is linear.*

Proof. Since the regularity of X means $X_p \neq 0$ for all $p \in M$, the assertion follows from proposition 2.1. □

Exercise 2.1. Prove that, if X is linear relative to (U, u), there is a chart (V, v) with $V \subseteq U$ in which (2.3) holds. (Hint: invert the proof of proposition 2.1.)

Exercise 2.2. Show that, if X is linear relative to (U, u), then it is regular on some open subset $V \subseteq U$, $X|_V \neq 0$. (Hint: the equation $C^i_j u^j = 0$ defines a $(\dim M - 1)$-dimensional submanifold of U (hence of M).)

If X is linear relative to a coordinate system $\{u^i\}$, there exist infinitely many other coordinates systems $\{v^i\}$ with respect to which it is linear too. Indeed, if, for some $C^j_i, B^j_i \in \mathbb{K}$,

$$X = (C^i_j u^j)\frac{\partial}{\partial u^i} = (B^i_j v^j)\frac{\partial}{\partial v^i}$$

on the intersection of the domains of $\{u^i\}$ and $\{v^i\}$, then

$$C^i_j u^j \frac{\partial v^k}{\partial u^i} = B^k_j v^j. \tag{2.4}$$

The problem is to be found v^i and B^j_i if u^i and C^j_i are given. For instance, if $\{u^i\} \mapsto \{v^i\}$ is linear, i.e. $v^i = a^i_j u^j$ with $a^j_i \in \mathbb{K}$ and $\det[a^j_i] \neq 0$, then the general solution of (2.4) is $[B^j_i] = [a^j_i] \cdot C \cdot [a^j_i]^{-1}$. However, equation (2.4) admits an infinite number of solutions $(\{v^i\}, \{B^i_j\})$ for which the change $\{u^i\} \mapsto \{v^i\}$ is nonlinear.

As a conclusion of the above results, we can say that the (local) regularity of a C^1 vector field is equivalent to its linearity relative to some local chart; such charts are not unique and their set is described via equation (2.4).

Suppose a vector field $X \in \mathfrak{X}(M)$ is linear relative to a chart (U, u) and (2.1) and (2.2) hold. The vector fields

$$E^j_i := u^j \frac{\partial}{\partial u^i} \in \mathfrak{X}(U) \tag{2.5}$$

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[1] Since we are interested only in the existence of the chart (U, u), we do not specify the functions f^i and constants a^i. For instance, the choice $a^1 = 1$, $f^1 = a \in \mathbb{K}$, $a^k = 0$ for $k \geq 2$, and $f^k = v^k$ for $k \geq 2$ results in the admissible coordinates $u^1 = a e^{v^1}$, $u^k = v^k$ for $k \geq 2$ and $U = V$.

form a basis on U for the vector fields linear relative to (U, u) in a sense that any such field X is on U a linear combination of E_i^j with *constant* coefficients, viz.

$$X|_U = \sum_{i,j} C_j^i E_i^j. \tag{2.6}$$

The vector fields (2.5) are generators of a Lie algebra with respect to the commutators as they are $\mathbb{K}$-linearly independent and

$$[E_j^i, E_l^k] = (\delta_j^k \delta_p^i \delta_l^q - \delta_l^i \delta_p^k \delta_j^q) E_q^p \tag{2.7}$$

where δ_i^j are the Kronecker deltas. Of course, the linear combinations $a_{jl}^{ik} E_k^l$, with $a_{jl}^{ik} \in \mathbb{K}$ and $E_j^i \mapsto a_{jl}^{ik} E_k^l$ invertible, form also a basis on U for the vector fields linear with respect to (U, u) and generate a Lie algebra.

Since the set $\{\frac{\partial}{\partial u^i}\}$ of linearly independent vector fields generates the module $\mathfrak{X}(M)$, the $(\dim M)^2$ vector fields $\{E_j^i\}$ are linearly dependent for $\dim M \geq 2$ and between them exist at least $(\dim M)^2 - \dim M$ (independent) connections.

3. Constant and affine vector fields

Similar to definition 2.1, we define the constant and affine vector field by

Definition 3.1. A vector field $X \in \mathfrak{X}(M)$ is termed *constant* or *affine* (*linear inhomogeneous*) relative to a chart (U, u) of M, if in the coordinate system $\{u^i\}$ associated with this chart, it has respectively the representation

$$X = B^i \frac{\partial}{\partial u^i} \tag{3.1}$$

$$X = (C_j^i u^j + B^i) \frac{\partial}{\partial u^i} \tag{3.2}$$

for some constant numbers $B^i, C_j^i \in \mathbb{K}$.

Obviously, the linear and constant vector fields are special case of the affine ones for $B^i \equiv 0$ or $C_i^j \equiv 0$, respectively.

Proposition 3.1 (cf. proposition 2.1)**.** *Let X be a C^1 vector field on a C^1 manifold M, $p \in M$, and $X_p \neq 0$. Then there exists charts (U, u) and (W, w) such that $U \ni p$, $W \ni p$ and X is constant relative to (U, u) and is affine with respect to (W, w).*

Proof. Let (V, v) be a chart of M such that $p \in V$ and (2.3) holds. Then, we can put $(U, u) = (V, v)$ and define (W, w) such that $W \subseteq V$ and assign to it coordinates functions to be $w^i = f^i(v^2, \ldots, v^{\dim M}) \mathrm{e}^{a^i v^1} + B^i v^1$, where the notation of the proof of proposition 2.1 is used and $B^i \in \mathbb{K}$ are constant numbers. $\square$

One can easily see (cf. (2.4)) that there exist infinitely many charts relative to which a vector field is constant/affine if it is constant/affine in a given chart; in particular, such coordinates systems for affine (constant) vector fields can be obtained from a given one via linear inhomogeneous transformations with *constant* coefficients (resp. via rescaling the coordinate functions by constants).

If (U, u) is a chart of M, the two sets of vector fields

$$E_i := \frac{\partial}{\partial u^i} \tag{3.3}$$

$$E_i^j := u^j \frac{\partial}{\partial u^i} \quad E_k = \frac{\partial}{\partial u^k} \tag{3.4}$$

form a basis for the vector fields which are respectively constant or affine relative to (U, u) in a sense that any such field is a linear combination with *constant* coefficients of the above fields. The elements of the former set generate an Abelian Lie algebra as $[E_i, E_j] = 0$, while the ones of the latter set generate a non-Abelian Lie algebra as their commutators are

$$[E_i, E_j] = 0 \tag{3.5a}$$
$$[E_i, E_k^j] = \delta_i^j E_k \tag{3.5b}$$
$$[E_j^i, E_l^k] = (\delta_j^k \delta_p^i \delta_l^q - \delta_l^i \delta_p^k \delta_j^q) E_q^p. \tag{3.5c}$$

4. Flows of vector fields

4.1. The 2-dimensional case

To begin with, consider a vector field X on $\mathbb{R}^2$ coordinated by (generally said, non-Cartesian) coordinates (u, v). We have the expansion $X = x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}$ in which the functions x and y are the components of X relative to the coordinate system (u, v). [2] We have $X(u) = x$ and $X(v) = y$. If $\beta_p \colon J \to \mathbb{R}^2$, with J being an open real interval, $0 \in J$ and $\beta_p(0) = p$ for some $p \in \mathbb{R}$, is a path along which X reduces to $\dot{\beta}$ (i.e. β is the integral path of X through p – *vide infra*), then the last equations reduce along β to

$$\frac{\mathrm{d}(u \circ \beta_p(t))}{\mathrm{d}t} = x \circ \beta_p(t) \quad \frac{\mathrm{d}(v \circ \beta_p(t))}{\mathrm{d}t} = y \circ \beta_p(t), \qquad t \in J. \tag{4.1}$$

The solutions of these equations with respect to $\beta_p(t)$ such that $\beta_p(0) = p$ for $p \in \mathbb{R}^2$ define the integral paths of X. They define a (local) 1-parameter group a of transformations of the $\mathbb{R}^2$ plane, termed also (local) flow of X, assigning to each $t \in J$ the mapping

$$\begin{aligned} &a_t \colon \mathbb{R}^2 \to \mathbb{R}^2 \\ &a_t \colon \mathbb{R}^2 \ni p \mapsto a_t(p) := \beta_p(t) \in \mathbb{R}^2 \end{aligned} \tag{4.2}$$

and locally represented via the mapping

$$(u_t, v_t) := (u, v) \circ a_t \circ (u, v)^{-1} \colon \mathbb{R}^2 \to \mathbb{R}^2. \tag{4.3}$$

So that, if a point $p \in \mathbb{R}^2$ has coordinates (p_u, p_v) in (u, v), i.e. $p = (u, v)^{-1}(p_u, p_v)$, then they change under the flow of X according to

$$(u_t, v_t) \colon (p_u, p_v) \mapsto (u, v) \circ a_t(p) =: (p_u(t), p_v(t)).$$

Under the flow a of a vector field X, the points in $\mathbb{R}^2$ move along their orbits (the integral paths of X) and any function $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ transforms into

$$f_t := f \circ a_t. \tag{4.4}$$

It is said that f is dragged along X (by the flow a) into f_t. If f is of class C^1, we have

$$X(f) = \frac{\mathrm{d}f_t}{\mathrm{d}t}\Big|_{t=0} = \lim_{t \to 0} \frac{f_t - f}{t}. \tag{4.5}$$

[2] More precisely, x and y are the components of X with respect to the natural frame $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$ induced by the coordinates (u, v).

4.2. The general case

Consider now the general case of a vector field $X \in \mathfrak{X}(M)$ on a C^1 manifold M.

In a local chart (U, u) of M with local coordinate system $\{u^1, \dots, u^{\dim M}\}$ is valid the expansion

$$X = X^i \frac{\partial}{\partial u^i} = \sum_{i=1}^{\dim M} X^i \frac{\partial}{\partial u^i} \tag{4.6}$$

in which $X^i \colon U \to \mathbb{K}$ are the components of X relative to the (natural) frame $\left\{\frac{\partial}{\partial u^i}\right\}$ on U. For a C^1 function $f \colon M \to \mathbb{K}$, we put

$$f_i := \frac{\partial f}{\partial u^i} \colon p \mapsto \frac{\partial f}{\partial u^i}\Big|_p := \frac{\partial (f \circ u^{-1})}{\partial r^i}\Big|_{u(p)} \tag{4.7}$$

with $\{r^1, \dots, r^{\dim M}\}$ being the standard Cartesian coordinate system on $\mathbb{K}^{\dim M}$, that is $r^i \colon \mathbb{K}^{\dim M} \ni (c_1, \dots, c_{\dim M}) \mapsto c_i$, so that $u^i = r^i \circ u$ and $r^1 = \mathsf{id}_{\mathbb{K}}$ for $\dim M = 1$. Therefore

$$X(u^i) = X^i. \tag{4.8}$$

Let J be an open $\mathbb{R}$-interval, $0 \in J$, and U be an open set in M. A *local 1-parameter group of local (C^k, if M is of class C^k) transformations* [3, ch. I, § 1] with domain $J \times U$ is a mapping $a \colon J \times U \to M$, $a \colon J \times U \ni (t, p) \mapsto a_t(p) \in M$, such that (i) for any $t \in J$, the mapping $a_t \colon U \ni p \mapsto a_t(p) \in a_t(U)$ is C^k diffeomorphism on $a_t(U)$ if M is of class C^k and (ii) if $s, t, s + t \in J$ and $p, a_t(p) \in U$, then

$$a_{s+t}(p) = a_s \circ a_t(p) \equiv a_s(a_t(p)). \tag{4.9}$$

If $U = M$ (and $J = \mathbb{R}$), a is called (global) 1-parameter group of transformations and $X \in \mathfrak{X}(M)$ is said to be complete in that case.

A local 1-parameter group a of transformations induces on U a vector field X via the equation $X_p = \dot\beta_p(0)$, where $p \in U$ and the path $\beta_p \colon J \to M$ is given by

$$\beta_p(t) := a_t(p), \tag{4.10}$$

i.e. β_p is the orbit of p and, at the same time, the integral path of X trough p.

The inverse is also true (see [3, ch. I, § 1, proposition 1.5]) or [2, theorem 1.48]). If $X \in \mathfrak{X}(M)$, for every $p \in M$ there exist an interval $J_p \ni 0$, open set U_p, and local 1-parameter group $a \colon J_p \times U_p \to M$ of local transformations which induces the given vector field X (on U_p). This local group is called *(local) flow* of X. In local coordinates it can be defined as follows.

Suppose (U, u) is a chart of M and $\{u^i = r^i \circ u\}$ is the corresponding local coordinate system. Recall that $\beta \colon J \to M$ is an integral path of X iff $X \circ \beta = \dot\beta$. Hence, if $X = X^i \frac{\partial}{\partial u^i}$, we get $X_{\beta(t)}(u^i) = (\dot\beta(t))(u^i) = \frac{\mathrm{d}(u^i \circ \beta(t))}{\mathrm{d}t}$. Combining the last result with (4.8), we find the equation of the integral paths of X as

$$\frac{\mathrm{d}(u^i \circ \beta(t))}{\mathrm{d}t} = X^i \circ \beta(t). \tag{4.11}$$

Let $\beta_p \colon J \to M$ be the integral path of X through a point $p \in M$, i.e. $\beta_p(0) = p$ and β_p satisfies (4.11) with β_p for β. Then the flow of X is given by $a_t(p) = \beta_p(t)$ and in (U, u) is represented by

$$u_t := u \circ a_t \circ u^{-1} \colon \mathbb{K}^{\dim M} \to \mathbb{K}^{\dim M} \tag{4.12}$$

and satisfies the system of differential equations

$$\frac{\mathrm{d}u_t^i(\lambda)}{\mathrm{d}t} = X^i \circ u^{-1} \circ u_t(\lambda) = X^i \circ u^{-1} \circ (u_t^1, \dots, u_t^{\dim M})(\lambda) \tag{4.13a}$$

for $i = 1, \dots, \dim M$ and $\lambda \in u(U)$, due to (4.11), and the initial condition

$$(u_t^1, \dots, u_t^{\dim M})|_{t=0} = \mathsf{id}_{\mathbb{K}^{\dim M}}. \tag{4.13b}$$

Equations (4.13a) simply mean that (4.8) are the local equations of the flow of X if $X^i = f^i(u^1, \dots, u^{\dim M})$ are known expressions of the local coordinate functions $u^1, \dots, u^{\dim M}$. Indeed, if this is the case, equations (4.8) along the integral paths of X reduce to (see (4.11) and (4.4))

$$\frac{\mathrm{d}u_t^i}{\mathrm{d}t} = f^i(u_t^1, \dots, u_t^{\dim M}), \tag{4.13a$'$}$$

which is an equivalent form of (4.13a)

Under the flow a of $X \in \mathfrak{X}(M)$, a function $f\colon M \to \mathbb{K}$ is dragged according to (4.4) and, if f is of class C^1, then its directional derivative with respect to X is given by (4.5).

4.3. Affine vector fields

Let us now find the flow of an affine vector field with local representation (3.2). For the purpose, we shall use the following matrix notation

$$\begin{aligned} &U := (u^1, \dots, u^{\dim M})^\top \quad \frac{\partial}{\partial U} := \Big(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^{\dim M}}\Big) \\ &C := [C^i_j] \quad B := (B^1, \dots, B^{\dim M})^\top \end{aligned} \tag{4.14}$$

in which the affine field (3.2) has the form [3]

$$X = \frac{\partial}{\partial U} \cdot (C \cdot U + B) \tag{4.15}$$

and hence

$$X(U) = C \cdot U + B. \tag{4.16}$$

In accordance with (4.12), the flow a of X is represented in the coordinates $\{u^i\}$ by the matrix $U_t = (u_t^1, \dots u_t^{\dim M})^\top$ with $u_t^i = u^i \circ a \circ ((u^1)^{-1}, \dots, (u^{\dim M})^{-1})$, i.e. $U_t = U \circ a_t \circ ((u^1)^{-1}, \dots, (u^{\dim M})^{-1})$. According to (5.1) and (4.16), U_t is the solution of the following matrix initial-valued problem:

$$\frac{\mathrm{d}U_t}{\mathrm{d}t} = C \cdot U_t + B \quad U_t|_{t=0} = U. \tag{4.17}$$

Therefore the explicit form of U_t is

$$U_t = tB + U \qquad \text{for } C = 0 \qquad \text{(constant field)} \tag{4.18a}$$

$$U_t = \mathrm{e}^{tC} U \qquad \text{for } B = 0 \qquad \text{(linear field)} \tag{4.18b}$$

$$U_t = \mathrm{e}^{tC}(U - \tilde{U}) + \tilde{U} \qquad \text{for } C \neq 0 \text{ and } B \neq 0 \text{ (affine field)}, \tag{4.18c}$$

where the constant matrix $\tilde{U}$ is a solution of the equation

$$C \cdot \tilde{U} + B = 0 \qquad \text{(with } C \neq 0 \text{ and } B \neq 0\text{)}. \tag{4.19}$$

(Note, the r.h.s. of (4.18c) is independent of the particular choice of $\tilde{U}$ if equation (4.19) has more than one solution with respect to $\tilde{U}$; if C is nondegenerate, then $\tilde{U} = -C^{-1} \cdot B$.) Thus the flow of a constant vector field is governed by a linear law while the one of a linear (affine) vector field is governed by an exponential law (combined with shift by $\tilde{U}$).

[3] At this point, we use the rule that summation excludes differentiation as a result of which the r.h.s. of (4.15) is simply a shortcut for the r.h.s. of (3.2).

5. Invariants and canonical parameters

Definition 5.1. A C^1 function $I\colon M \to \mathbb{K}$ is called an *invariant* of $X \in \mathfrak{X}(M)$ if it is constant along the integral paths of X, $I_t = I = \text{const}$ for all t or $X(I) = 0$. A C^1 function $S\colon M \to \mathbb{K}$ is termed a *canonical parameter* of X if $X(S) = 1$.

The following result is completely obvious but worth recording.

Proposition 5.1. *The difference of two canonical parameters of X is an invariant of X and the sum of a canonical parameter and invariant of X is a canonical parameter of X. Consequently, a canonical parameter is defined up to an invariant.*

Any constant function $M \to \{c\}$ for a given $c \in \mathbb{K}$ is an invariant of all vector fields on M. However, the existence of non-trivial invariants as well as of canonical parameters is not evident. Generally they exist only locally as stated in the following result.

Proposition 5.2. *Given a point $p \in M$ and C^1 vector field on a C^3 real manifold M (or C^3 complex manifold considered as a real one of real dimension $\dim_{\mathbb{R}} M = 2\dim_{\mathbb{C}} M$) such that $X_p \neq 0$. Then there is an open set $V \subseteq M$ containing p, $V \ni p$, on which exist a canonical parameter S and non-constant invariant I of X.*

Proof. According to [2, proposition 1.53], there is a local chart (V, v) with coordinate functions v^i such that $V \ni p$ and

$$X|_V = \frac{\partial}{\partial v^1}. \tag{5.1}$$

Defining

$$S = v^1 + F(v^2, \dots, v^{\dim M}) \tag{5.2a}$$

$$I = G(v^2, \dots, v^{\dim M}), \tag{5.2b}$$

where F and G map a $(\dim M - 1)$-tuple of C^1 functions on V into a C^1 function on V, from (5.1), we see that S and I are respectively a canonical parameter and an invariant of X on V, i.e. $X|_V(S) = 1$ and $X|_V(I) = 0$. □

Corollary 5.1. *Under the hypotheses of proposition 5.2 and the notation introduce in its proof, all local canonical parameters and invariants of a vector field are given via* (5.2a) *and* (5.2b)*, respectively.*

Proof. Use proposition 5.1 and the proof of proposition 5.2. □

Exercise 5.1. If $\dim M = 1$, prove that the only ($\mathbb{K}$-valued) invariants of a vector field, with separable (by open sets) points at with it is irregular, if any, are the constant functions $M \to \{c\}$ for some $c \in \mathbb{K}$.

In a neighborhood of a point p at which $X_p \neq 0$, the proof of proposition 5.2 provides the canonical parameter v^1 and, if $\dim M \geq 2$, the $n-1$ invariants $v^2, \dots, v^{\dim M}$. The set of these functions $\{v^i\}$ is a coordinate system on V in which (5.1) holds. The converse of that observation reads

Proposition 5.3. *Let $X \in \mathfrak{X}(M)$, S be a canonical parameter of X, and $I^2 \dots, I^{\dim M}$ be invariants of X on $U \subseteq M$. If the set $\{S, I^2 \dots, I^{\dim M}\}$ is a coordinate system on U, i.e. for some chart (U, u) of M, then in it*

$$X|_U = \frac{\partial}{\partial S} \tag{5.3}$$

and the flow a of X in it is represented via the mapping $u \mapsto u_t = (r^1 + t, r^2, \dots, r^{\dim M})$ (see (4.12)*) or*

$$u_t \colon (s, \boldsymbol{i}) \mapsto (s_t, \boldsymbol{i}_t) = (s + t, \boldsymbol{i}) \tag{5.4}$$

for all $s \in \mathbb{K}$ and $\boldsymbol{i} \in \mathbb{K}^{\dim M - 1}$.

Proof. Let $\{u^i\}$ be arbitrary coordinate system on U and $X|_U = X^i \frac{\partial}{\partial u^i}$. Making the change $\{u^i\} \mapsto \{S, I^1 \dots, I^{\dim M}\}$, we get

$$X|_U = X^i \frac{\partial}{\partial u^i} = X^i \Big\{ \frac{\partial S}{\partial u^i} \frac{\partial}{\partial S} + \sum_{k=2}^{\dim M} \frac{\partial I^k}{\partial u^i} \frac{\partial}{\partial I^k} \Big\} = X(S) \frac{\partial}{\partial S} + \sum_{k=2}^{\dim M} X(I^k) \frac{\partial}{\partial I^k} = \frac{\partial}{\partial S},$$

where $X(f) = f_i X^i$, for a C^1 function f, and definition 5.1 were applied. To prove (5.4), we notice that, by virtue of (5.3), the equations (4.13a) in $\{S, I^1 \dots, I^{\dim M}\}$ read

$$\frac{\mathrm{d} u_t^1(\lambda)}{\mathrm{d} t} = 1 \quad \frac{\mathrm{d} u_t^k(\lambda)}{\mathrm{d} t} = 0 \text{ for } k \geq 2$$

and their solution $u_t = (u_t^1, \dots, u_t^{\dim M})$, under the condition (4.13b), is $u_t^1 = r^1 + t$ and $u_t^k = r^k$ for $k > 1$, where $\{r^i\}$ is the standard coordinate system on $\mathbb{K}^{\dim M}$ and t is considered as the constant function $\mathbb{K}^{\dim M} \to t$. □

Example 5.1. Consider $\mathbb{R}^2$ coordinated by the standard Cartesian coordinates $(u, v) = (r^1, r^2)$ and the affine vector field $X = \alpha \frac{\partial}{\partial u} + (2\beta u + \gamma) \frac{\partial}{\partial v}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha \neq 0$. The flow a of X is locally represented by (see (4.12)) $(u_t, v_t) := (u, v) \circ a_t \circ (u, v)^{-1} \colon \mathbb{R}^2 \to \mathbb{R}^2$ and is the solution of the initial-value problem (see (4.13))

$$\begin{aligned} &\frac{\mathrm{d} u_t}{\mathrm{d} t} = \alpha \quad \frac{\mathrm{d} v_t}{\mathrm{d} t} = (2\beta u + \gamma) \circ (u, v)^{-1} \circ (u_t, v_t) = 2\beta u_t + \gamma, \\ &(u_0, v_0) = (r^1, r^2) = \mathsf{id}_{\mathbb{K}^2}, \end{aligned}$$

so that $u_t = u + \alpha t$ and $v_t = v + (2\beta u + \gamma) t + \alpha \beta t^2$, i.e.

$$(u_t, v_t) \colon \mathbb{R} \ni (b, c) \mapsto (b_t, c_t) = (b + \alpha t, c + (2\beta b + \gamma) t + \alpha \beta t^2).$$

From here and (4.10), we see that the local coordinates of the point $\beta_p(t)$ of the integral path of X passing through $p = (u, v)^{-1}(b, c)$ are

$$(u, v)(\beta_p(t)) = (b_t, c_t) = (b + \alpha t, c + (2\beta b + \gamma) t + \alpha \beta t^2)$$

The particular vector field X has a (global) canonical parameter $S = \frac{1}{\alpha} u$ and invariant $I = \alpha v - \beta u^2 - \gamma$, $X(S) = 1$ and $X(I) = 0$. This canonical parameter and invariant (as well as all of them) can be found in a way similar to the one described below in example 5.2.

Applying corollary 5.1, we can assert that all canonical parameters of X are $S + F(I)$ and all its invariants are $G(I)$, where F and G map C^1 functions on $\mathbb{R}^2$ into C^1 functions on $\mathbb{R}^2$

The Jacobian of the coordinate change $(u, v) \mapsto (S, I)$ is $\begin{vmatrix} 1/\alpha & 0 \\ -2\beta u & \alpha \end{vmatrix} = 1 \neq 0$. Therefore the pair (S, I) induces the natural frame $\{\frac{\partial}{\partial S}, \frac{\partial}{\partial I}\}$ on the set $U = \mathbb{R}^2$ and, by proposition 5.3, on U in (S, I) we have

$$X|_U = \frac{\partial}{\partial S} \qquad (S_t, I_t) \colon U \ni (b, c) \mapsto (b_t, c_t) = (b + t, c).$$

△

Example 5.2. Consider a constant vector field X with nonvanishing coefficients,

$$X = B^i \frac{\partial}{\partial u^i} \in \mathfrak{X}(M) \qquad B^i \neq 0 \text{ for all } i.$$

The C^1 functions S and I are respectively a canonical parameter and an invariant of X iff they are solutions of the differential equations

$$B^i \frac{\partial S}{\partial u^i} = 1, \quad B^i \frac{\partial I}{\partial u^i} = 0$$

According to the general theory of differential equations of this kind [4, pp. 733–735], these equations, relative to S and I, are equivalent to respectively the systems

$$\begin{aligned} \frac{\mathrm{d}u^1}{B^1} &= \cdots = \frac{\mathrm{d}u^{\dim M}}{B^{\dim M}} = \frac{\mathrm{d}S}{1} \\ \frac{\mathrm{d}u^1}{B^1} &= \cdots = \frac{\mathrm{d}u^{\dim M}}{B^{\dim M}} \quad \mathrm{d}I = 0 \end{aligned}$$

and, consequently, the general form of the functions S and I is determined via the equations $\Phi(\varphi_1, \dots, \varphi_{\dim M}) = 0$ and $\Psi(\psi_1, \dots, \psi_{\dim M}) = 0$, where Φ and Ψ are arbitrary C^1 functions and

$$\begin{aligned} &\varphi_1 = S - \frac{u^1}{B^1} \quad && \varphi_i = u^i - u^1 \frac{B^i}{B^1} \text{ for } i \geq 2 \\ &\psi_1 = I && \psi_i = u^i - u^1 \frac{B^i}{B^1} \text{ for } i \geq 2 \end{aligned}$$

are n independent integrals of the above systems. Admitting that the last equations can be solved with respect to S and I, we get

$$\begin{aligned} S &= \frac{u^1}{B^1} + F\Big(u^2 - u^1 \frac{B^2}{B^1}, \dots, u^{\dim M} - u^1 \frac{B^{\dim M}}{B^1}\Big) \\ I &= G\Big(u^2 - u^1 \frac{B^2}{B^1}, \dots, u^{\dim M} - u^1 \frac{B^{\dim M}}{B^1}\Big) \end{aligned}$$

for some C^1 functions F and G, which agrees with corollary 5.1. △

Remark 5.1. If some of the B's in (3.1) vanish, the procedure described in example 5.2 remains valid for the remaining non-vanishing B's and to the obtained in this way expressions for S and I can be added arbitrary functions of the u's for which the similarly indexed B's vanish.

In a way similar to the one considered in example 5.2, one can investigate the problems of finding the canonical parameters and invariants of linear and affine vector fields. The only difference from the case of a constant vector field is in the more difficult differential equations that should be solved.

6. Fundamental vector fields

A left (right) action of a Lie group G, with multiplication $G \times G \ni (a, b) \mapsto ab \in G$ and identity element e, on a manifold M is a mapping $\lambda\colon G \times M \to M$ ($\rho\colon M \times G \to M$) such that $\lambda_{ab} = \lambda_a \circ \lambda_b$ and $\lambda_e = \mathsf{id}_M$ ($\rho^{ab} = \rho^b \circ \rho^a$ and $\rho^e = \mathsf{id}_M$) for $a, b \in G$, where the partial mappings $\lambda_a, \rho^a\colon M \to M$ are defined by $\lambda_a(x) := \lambda(a, x)$ and $\rho^a(x) := \rho(x, a)$ for all $x \in M$. Below we shall need also the partial mappings $\lambda^x, \rho_x\colon G \to G$ defined by $\lambda^x(a) := \lambda(a, x)$ and $\rho_x(a) := \rho(x, a)$ for all $x \in M$ and $a \in G$. A fundamental vector field on M is a vector field on M which is obtained from a vector in the space $T_e(G)$ tangent to G at the identity element $e \in G$ via the tangent mappings of λ^x or ρ_x. Precisely, we have following definition.

Definition 6.1. Let λ and ρ be respectively left and right actions of a C^1 Lie G on M. For $X \in T_e(G)$, the *left and right fundamental vector fields* $\xi^l_X \in \mathfrak{X}(M)$ and $\xi^r_X \in \mathfrak{X}(M)$ on M (associated with X and the given actions of G) are defined respectively by

$$\xi^l_X \colon x \mapsto \xi^l_X(x) := (T_e(\lambda^x))(X) = (T_{(e,x)}(\lambda))(X, 0_x) \in T_{\lambda(e,x)}(M) = T_x(M)$$
$$\xi^r_X \colon x \mapsto \xi^r_X(x) := (T_e(\rho_x))(X) = (T_{(x,e)}(\rho))(0_x, X) \in T_{\rho(x,e)}(M) = T_x(M)$$

where $0_x \in T_x(M)$ is the zero vector in the space $T_x(M)$ tangent to M at $X \in M$.

The above definitions can be localized in an evident way if we replace in them M by an arbitrary open subset $V \subseteq M$.

Obviously, the vectors $\xi^l_X(x)$ and $\xi^r_X(x)$ are tangent at x to the orbits $\lambda^x(G)$ and $\rho_x(G)$, respectively, of G through x.

More information on fundamental vector fields can be found, for instance, in [5, pp. 46–47], [6, pp. 283–284], [7, pp. 121–124] and [3]. It should be noted that sometimes, e.g. in [8], [9] and [10, see especially pp. 127–133], the term 'operator of a group' is used instead of the modern one 'fundamental vector field' associated to a group (action on a manifold).

Since the sets of left and right invariant vector fields, [4] on G are isomorphic as vector spaces to $T_e(G)$ [5] the left (resp. right) fundamental vector fields are images of the left or right invariant vector fields on G via the left (resp. right) action of the group; [6] note that a *left/right* fundamental vector fields are connected with the *left/right action of the group* and not with the left or right invariant vector field from which one has started. [7] Consequently the fundamental vector fields are images of the Lie algebra $\mathfrak{g}$ of G via the group actions as $\mathfrak{g}$ is identified with the set of left invariant vector fields on G [3], or with the set of right invariant vector fields on G [1, p. 42, definition 1.44], or with the space $T_e(G)$ tangent to G at the identity element e [5].

If $\{y^\mu : \mu = 1, \dots, \dim G\}$ are local coordinates on an open subset $U \subseteq G$ and $\{u^k : k = 1, \dots, \dim M\}$ are coordinates on M with $\lambda^x(U)$ in their domain, then [2, sec. 1.23(a)]

$$(T_a(\lambda^x))\Big(\frac{\partial}{\partial y^\mu}\Big|_a\Big) = \frac{\partial(u^k \circ \lambda^x)}{\partial y^\mu}\Big|_a \frac{\partial}{\partial u^k}\Big|_{\lambda^x(a)} \qquad a \in G; \tag{6.1}$$

we have a similar equation with ρ_x for λ^x in a case of a right action ρ when $\rho_x(U)$ is in the domain of $\{u^k\}$. If $e \in U$ and $X = X^\mu \frac{\partial}{\partial y^\mu}\big|_e$, then the last equation immediately implies

$$\xi^l_X(x) = X^\mu \frac{\partial(u^k \circ \lambda^x)}{\partial y^\mu}\Big|_e \frac{\partial}{\partial u^k}\Big|_x \tag{6.2a}$$
$$\xi^r_X(x) = X^\mu \frac{\partial(u^k \circ \rho_x)}{\partial y^\mu}\Big|_e \frac{\partial}{\partial u^k}\Big|_x . \tag{6.2b}$$

[4] If $L_a \colon G \ni b \mapsto ab$ and $R_a \colon G \ni b \to ba$ are the left and right translations on G by an element $a \in G$, a vector field $X \in \mathfrak{X}(G)$ is left or right invariant if $(L_a)_*(X_b) = X_{L_a b} = X_{ab}$ or $(R_a)_*(X_b) = X_{R_a b} = X_{ba}$, respectively.

[5] If $X \in \mathfrak{X}(G)$ is left or right invariant, the isomorphism $I \colon \{X\} \to T_e(G)$ is given by $I \colon X \mapsto X_e$, sending X to its value at the identity e, as $X_a = X_{L_e(a)} = (L_a)_*(X_e)$ or $X_a = X_{R_e(a)} = (R_a)_*(X_e)$, respectively.

[6] Indeed, if Y is a left or right invariant vector field on G, the left action $\lambda \colon G \times M \to M$ sends it into a vector field $Y^l \in \mathfrak{X}(M)$ such that $Y^l \colon X \mapsto Y^l(x) := Y^l(\lambda(e, x)) = (T_e(\lambda^x))(Y|_e)$, which depends only on Y at the identity e of G. Similarly, a right action $\rho \colon M \times G \to M$ sends Y to $Y^r \in \mathfrak{X}(M)$ such that $Y^r \colon x \mapsto Y^r(x) = (T_e(\rho_x))(Y|_e)$.

[7] The reason for that being the isomorphism I described in footnote 5 above: $\xi^l_{I(X)}(x) = T_e(\lambda^x)(X_e)$ or $\xi^r_{I(X)}(x) = T_e(\rho_x)(X_e)$ regardless is $X \in \mathfrak{X}(M)$ left or right invariant.

6.1. Fundamental vector fields on $\mathbb{R}^n$ induced by $\mathrm{GL}(n,\mathbb{R})$

Let the general linear group $\mathrm{GL}(n,\mathbb{R})$, consisting of all regular $n\times n$, $n\in\mathbb{N}$, matrices with matrix multiplication as a group multiplication and the identity $n\times n$ matrix $\mathbb{1}$ as an identity element, be represented on $\mathbb{R}^n$ via a left action $\lambda\colon \mathrm{GL}(n,\mathbb{R})\times\mathbb{R}^n\to\mathbb{R}^n$ given by left multiplication, viz., if $a=[a^i_j]\in\mathrm{GL}(n,\mathbb{R})$ and $x=(x^1,\dots,x^n)^\top\in\mathbb{R}^n$, where $\top$ means matrix transposition, then $\lambda_a(x)=\lambda^x(a)=\lambda(a,x)=a\cdot x=(a^1_jx^j,\dots,a^n_jx^j)^\top$.

Proposition 6.1. *The vector field*

$$\xi_X\colon x\mapsto\xi_X(x):=T_{\mathbb{1}}(\lambda^x)(X)=\sum_{i,j}X^i_ju^j(x)\frac{\partial}{\partial u^i}\Big|_x\in T_x(\mathbb{R}^n) \tag{6.3}$$

is a (left) fundamental vector field on $\mathbb{R}^n$ corresponding to $X\in T_{\mathbb{1}}(\mathrm{GL}(n,\mathbb{R}))$ and the representiion λ of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$ by left multiplication. Here X^j_i are the components of X in the coordinate system defined a little below in the proof of this assertion.

Proof. Consider the tangent mapping $T_a(\lambda^x)\colon T_a(G)\to T_{\lambda^x(a)}(\mathbb{R}^n)=T_{a\cdot x}(\mathbb{R}^n)$. In the global coordinate systems $\{y^i_j\}$ on $\mathrm{GL}(n,\mathbb{R})$ and $\{u^i\}$ on $\mathbb{R}^n$, given by $y^i_j(a):=a^i_j$ and $u^i(x):=x^i$, it is represented by a Jacobi matrix with elements $\frac{\partial(u^k\circ\lambda^x)}{\partial y^i_j}$ such that

$$(T_a(\lambda^x))\Big(\frac{\partial}{\partial y^i_j}\Big|_a\Big)=\frac{\partial(u^k\circ\lambda^x)}{\partial y^i_j}\Big|_a\frac{\partial}{\partial u^k}\Big|_{\lambda^x(a)}$$

due to (6.1). Since

$$u^k\circ\lambda^x\colon a\mapsto u^k(\lambda^x(a))=u^k(a\cdot x)=a^k_jx^j=u^j(x)y^k_j(a) \tag{6.4}$$

the elements of the Jacobi matrix are $u^j(x)\delta^k_i$, so that

$$(T_a(\lambda^x))\Big(\frac{\partial}{\partial y^i_j}\Big|_a\Big)=u^j(x)\frac{\partial}{\partial u^i}\Big|_{\lambda^x(a)}=x^j\frac{\partial}{\partial u^i}\Big|_{a\cdot x} \tag{6.5}$$

and, if $Y_a=(Y_a)^i_j\frac{\partial}{\partial y^i_j}\big|_a$, then

$$(T_a(\lambda^x))(Y_a)=(Y_a)^i_ju^j(x)\frac{\partial}{\partial u^i}\Big|_{\lambda^x(a)}=(Y_a)^i_jx^j\frac{\partial}{\partial u^i}\Big|_{a\cdot x}. \tag{6.6}$$

The particular settings $a=\mathbb{1}=[\delta^i_j]$ and $X=Y_{\mathbb{1}}\in T_{\mathbb{1}}(\mathrm{GL}(n,\mathbb{R}))$ in the last equation reduce it to

$$(T_{\mathbb{1}}(\lambda^x))(X)=X^i_ju^j(x)\frac{\partial}{\partial u^i}\Big|_x \tag{6.7}$$

from where (6.3) follows. □

Evidently, the fundamental vector field

$$\xi_X=X^i_ju^j\frac{\partial}{\partial u^i}\in\mathfrak{X}(\mathbb{R}^n) \tag{6.8}$$

is a linear vector field (in the frame/coordinates used above) to which corresponds the matrix $C=[X^i_j]$. In particular, to the vectors in $T_{\mathbb{1}}(\mathrm{GL}(n,\mathbb{R}))$ with components $X^k_l=\delta^i_l\delta^k_j$ in $\{y^i_j\}$ correspond the fundamental vector fields

$$E^i_j=u^i\frac{\partial}{\partial u^j}\in\mathfrak{X}(\mathbb{R}^n) \tag{6.9}$$

which form a basis for the set of linear vector fields in the sense that any linear vector field is a linear combination with *constant* coefficients of these vector fields. These vector fields are generally linearly dependent and between them exist at least n^2-n (independent) connections.

Corollary 6.1. *All linear (relative to Cartesian coordinates) vector fields on $\mathbb{R}^n$ are fundamental vector fields of* $\mathrm{GL}(n,\mathbb{R})$ *represented on $\mathbb{R}^n$ via left multiplication and* vice versa.

Proof. If $c^i_j u^j \frac{\partial}{\partial u^j} \in \mathfrak{X}(\mathbb{R})^n$ is a linear vector field in the Cartesian coordinate system $\{u^i\}$, then, by equation (6.8), it is the fundamental vector field corresponding to the vector $X = c^i_j \frac{\partial}{\partial y^i_j} \in T_{\mathbb{1}}(\mathrm{GL}(n,\mathbb{R}))$. The converse was proved above. □

Remark 6.1. A vector field on $\mathbb{R}^n$ which is linear relative to non-Cartesian coordinates need not to be a fundamental vector field for $\mathrm{GL}(n,\mathbb{R})$. For instance, if $n=1$ and u is the standard Cartesian coordinate on $\mathbb{R}$, the field $X = \frac{u}{u+1}\frac{\mathrm{d}}{\mathrm{d}u}$ is non-linear in $\{u\}$ and, consequently, is non-fundamental for $\mathrm{GL}(1,\mathbb{R})$, but in a non-Cartesian coordinate system with the coordinate function $t = ue^u$ it has the representation $X = t\frac{\mathrm{d}}{\mathrm{d}t}$ and hence it is linear in $\{t\}$. The choice of the coordinates $\{y^j_i\}$ on $\mathrm{GL}(n,\mathbb{R})$ is not so important. According to (6.4), their change results in replacing x^j in (6.5) (or u^j in or after it) with $A^j_i x^i$ (resp. $A^j_i u^i$) where A^j_i are constants depending only on the matrix a. Similar is the result if instead of the standard Cartesian coordinates $\{u^i\}$ on $\mathbb{R}^n$ one uses any set of non-standard Cartesian coordinates on $\mathbb{R}^n$.

Similar remarks hold true with respect to Corollaries 6.2 and 6.3 below.

6.2. Fundamental vector fields on $\mathbb{R}^n$ induced by T_n

As a set, the translation group T_n on $\mathbb{R}^n$ coincides with $\mathbb{R}^n$, $\mathrm{T}_n = \mathbb{R}^n$, with addition as a group multiplication and the zero vector as identity element; hence T_n is an Abelian group. Its left and right actions $\lambda\colon \mathrm{T}_n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\rho\colon \mathbb{R}^n \times \mathrm{T}_n \to \mathbb{R}^n$, respectively, on $\mathbb{R}^n$ are defined by $\lambda(t,x) = \rho(x,t) = x + t$ for all $t \in \mathrm{T}_n$ and $x \in \mathbb{R}^n$. Let $\{z^i\}$ and $\{u^i\}$ be coordinate systems on respectively T_n and $\mathbb{R}^n$ such that $z^i(t) = t^i$ and $u^i(x) = x^i$ for $t = (t^1,\dots,t^n)^\top \in \mathrm{T}_n$ and $x = (x^1,\dots,x^n)^\top \in \mathbb{R}^n$.

Proposition 6.2. *The left and right fundamental vector fields for T_n coincide and the fundamental vector field associated with $X = X^i \frac{\partial}{\partial z^i}\big|_{\mathbf{0}} \in T_{\mathbf{0}}(\mathrm{T}_n)$, $\mathbf{0}$ being the zero vector of T_n (i.e. of $\mathbb{R}^n$), is*

$$\xi_X = X^i \frac{\partial}{\partial u^i} \in \mathfrak{X}(\mathbb{R}^n). \tag{6.10}$$

Proof. One can easily prove that $u^k \circ \lambda^x = u^k \circ \rho_x = x^k + z^k$ and

$$(T_t(\lambda^x))\Big(\frac{\partial}{\partial z^i}\Big|_t\Big) = (T_t(\rho_x))\Big(\frac{\partial}{\partial z^i}\Big|_t\Big) = \frac{\partial}{\partial u^i}\Big|_{x+t}. \tag{6.11}$$

Therefore

$$(T_t(\lambda^x))(Y_t) = (T_t(\lambda^x))(Y_t) = (Y_t)^i \frac{\partial}{\partial u^i}\Big|_{x+t} \tag{6.12}$$

for $Y_t = (Y_t)^i \frac{\partial}{\partial z^i}\big|_t \in T_t(\mathrm{T}_n)$. Putting here $t = \mathbf{0}$ and $Y_{\mathbf{0}} = X$, we get (6.10). □

Obviously, the fundamental vector fields

$$E_i = \frac{\partial}{\partial u^i} \in \mathfrak{X}(\mathbb{R}^n) \tag{6.13}$$

form a basis for the set of fundamental fields of T_n in a sense that any such field is their linear combination with *constant* coefficients.

The following result is almost trivial but nevertheless worth recording.

Corollary 6.2. *A vector field on $\mathbb{R}^n$ is with constant components (relative to Cartesian coordinates) iff it is a fundamental vector field of T_n.*

6.3. Fundamental vector fields on $\mathbb{R}^n$ induced by $\mathrm{GA}(n,\mathbb{R})$

The general affine group $\mathrm{GA}(n,\mathbb{R})$ is a semidirect product (sum) of the general linear group $\mathrm{GL}(n,\mathbb{R})$ and the translation group T_n, $\mathrm{GA}(n,\mathbb{R}) = \mathrm{GL}(n,\mathbb{R}) \rtimes \mathrm{T}_n$. [8] If $a_1, a_2 \in \mathrm{GL}(n,\mathbb{R})$ and $t_1, t_2 \in \mathrm{T}_n$, the product of the elements $a_1 \rtimes t_1$ and $a_2 \rtimes t_2$ in $\mathrm{GA}(n,\mathbb{R})$ is $(a_1 \rtimes t_1)(a_2 \rtimes t_2) := (a_1 \cdot a_2) \rtimes (a_1 \cdot t_2 + t_1)$. A left action $\lambda\colon\ \mathrm{GA}(n,\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\lambda(a \rtimes t, x) := a \cdot x + t \tag{6.14}$$

for all $a = [a_i^j] \in \mathrm{GL}(n,\mathbb{R})$, $t = (t^1, \dots, t^n)^\top \in \mathrm{T}_n$ and $x \in \mathbb{R}^n$.

Proposition 6.3. *The fundamentals vector field ξ_X on $\mathbb{R}^n$ corresponding to $X = X_j^i \frac{\partial}{\partial y_j^i}\big|_{\mathbb{1}\rtimes\mathbf{0}} + X^i \frac{\partial}{\partial z^i}\big|_{\mathbb{1}\rtimes\mathbf{0}} \in T_{\mathbb{1}\rtimes\mathbf{0}}(\mathrm{GA}(n,\mathbb{R}))$ and the left action λ, given by* (6.14)*, is*

$$\xi_X = (X_j^i u^j + X^i) \frac{\partial}{\partial u^i} \in \mathfrak{X}(\mathbb{R}^n) \tag{6.15}$$

in the coordinate systems $\{y_j^i, z^k\}$ and $\{u^i\}$ defined on the next lines.

Proof. Define global coordinate systems $\{y_j^i, z^k\}$ on $\mathrm{GA}(n,\mathbb{R})$ and $\{u^i\}$ on $\mathbb{R}^n$ by $y_j^i(a \rtimes t) := a_j^i$, $z^i(a \rtimes t) := t^i$ and $u^k(x) = x^k$. Then $u^k \circ \lambda^x = u^l(x) y_l^k + z^k$ and consequently

$$(T_{a\rtimes t}(\lambda^x))\Big(Y_j^i \frac{\partial}{\partial y_j^i}\Big|_{a\rtimes t} + Z^i \frac{\partial}{\partial z^i}\Big|_{a\rtimes t}\Big) = (Y_j^i u^j(x) + Z^i) \frac{\partial}{\partial u^i}\Big|_{a\cdot x+t} \in T_{a\cdot x+t}(\mathbb{R}^n) \tag{6.16}$$

for all $Y_j^i, Z^i \in \mathbb{R}$. The assertion now follows from here and definition 6.1. □

Corollary 6.3. *A vector field on $\mathbb{R}^n$ is an affine vector field (relative to some Cartesian coordinates) iff it is a fundamental vector field of $\mathrm{GA}(n,\mathbb{R})$ represented on $\mathbb{R}^n$ via the left action λ described above.*

6.4. Local left actions of $\mathrm{GA}(n,\mathbb{K})$ on a manifold

The general affine group $\mathrm{GA}(n,\mathbb{K})$ and its subgroups $\mathrm{GL}(n,\mathbb{K})$ and T_n have natural (local) left actions on an arbitrary manifold M of dimension $\dim M = n$.

Let λ be the left action described in subsection 6.3 with $\mathbb{R}$ replaced with $\mathbb{K}$, $x \in M$ and (V, v) be a chart of M with x in its domain, $x \in V$, and coordinate diffeomorphism $v\colon V \to \mathbb{K}^n$. Define a mapping $L\colon\ \mathrm{GA}(n,\mathbb{K}) \times V \to V$ by

$$L(a \rtimes t, x) := v^{-1}(a \cdot v(x) + t) = v^{-1} \circ \lambda(a \rtimes t, v(x)) \tag{6.17}$$

for all $a \rtimes t \in \mathrm{GA}(n,\mathbb{K})$, where $v(x) \in \mathbb{K}^n$ is considered as a vector-column. We have $L(a \rtimes t, x) \in V$ as $v(V) = \mathbb{K}^n$. Since

$$L^x := L(\cdot, x) = v^{-1} \circ \lambda^{v(x)} \quad L_{a\rtimes t} := L(a \rtimes t, \cdot) = v^{-1} \circ \lambda_{a\rtimes t} \circ v \tag{6.18}$$

and $\lambda\colon\ \mathrm{GA}(n,\mathbb{K}) \times \mathbb{K}^n \to \mathbb{K}^n$ is a left action (see (6.14)), the mapping L is a *local* left action of $\mathrm{GA}(n,\mathbb{K})$ on $V \subseteq M$, which, obviously, depends on the chart (V, v).

The (local) fundamental vector field $\xi_X \in \mathfrak{X}(M)$ corresponding to $X \in T_{\mathbb{1}\rtimes\mathbf{0}}(\mathrm{GA}(n,\mathbb{K}))$ and L is describe by the following proposition.

[8] For a matrix realization of $\mathrm{GA}(n,\mathbb{R})$ as a subgroup of $\mathrm{GL}(n+1,\mathbb{R})$, see [3, ch. III, § 3].

Proposition 6.4. *The local fundamental vector field* $\xi_X \in \mathfrak{X}(M)$ *corresponding to* $X = X^i_j \frac{\partial}{\partial y^i_j}\big|_{1\!\!1\rtimes\mathbf{0}} + X^i \frac{\partial}{\partial z^i}\big|_{1\!\!1\rtimes\mathbf{0}} \in T_{1\!\!1\rtimes\mathbf{0}}(\mathrm{GA}(n,\mathbb{R}))$, *where* $\{y^j_i, z^k\}$ *are the coordinates on* $\mathrm{GA}(n,\mathbb{K})$ *introduced in subsection 6.3, and* L *is*

$$\xi_X = (X^i_j v^j + X^i)\frac{\partial}{\partial v^i} \in \mathfrak{X}(V) \subseteq \mathfrak{X}(M). \tag{6.19}$$

Proof. Let $\{u^i\}$ be the standard Cartesian coordinate system on $\mathbb{K}^n$ and $\{v^i\}$ be the coordinate system defined by the chart (V, v). Since $u^i \circ v := v^i$, we have

$$v^k \circ L^x = v^k \circ v^{-1} \circ \lambda^{v(x)} = u^k \circ \lambda^{v(x)} = u^l(v(x))y^k_l + z^k = v^l(x)y^k_l + z^k. \tag{6.20}$$

Therefore

$$(T_{a\rtimes t}(L^x))\Big(Y^i_j \frac{\partial}{\partial y^i_j}\Big|_{a\rtimes t} + Z^i \frac{\partial}{\partial z^i}\Big|_{a\rtimes t}\Big) = (Y^i_j v^j(x) + Z^i)\frac{\partial}{\partial v^i}\Big|_{L(a\rtimes t,x)} \in T_{L(a\rtimes t,x)}(M) \tag{6.21}$$

for all $Y^i_j, Z^i \in \mathbb{K}$. The assertion now follows from here and definition 6.1. □

The following corollary is evident.

Corollary 6.4. *A vector field on a* C^1 *(real or complex) manifold* M *is an affine vector field relative to a chart* (V, v) *if and only if it reduces on* V *to a fundamental vector field of the general affine group* $\mathrm{GA}(n,\mathbb{K})$, $n = \dim M$, *represented on* M, *precisely on* V, *via the left action* L *defined by* (6.17).

Since the groups $\mathrm{GL}(n,\mathbb{K})$ and T_n are subgroups of $\mathrm{GA}(n,\mathbb{K})$, the above considerations can be applied *mutatis mutandis* to them. Without going into details, this can be done as follows.

The local left action of $\mathrm{GL}(n,\mathbb{K})$ and T_n on M in (V, v) are given respectively by (cf. (6.17))

$$L\colon \mathrm{GL}(n,\mathbb{K}) \times V \to V\colon (a, x) \mapsto v^{-1}(a \cdot v(x)) \tag{6.22a}$$
$$L\colon \mathrm{T}_n \times V \to V\colon (t, x) \mapsto v^{-1}(v(x) + t). \tag{6.22b}$$

Proposition 6.5. *The fundamental vector fields corresponding to* $X = X^i_j \frac{\partial}{\partial y^i_j} \in T_{1\!\!1}(\mathrm{GL}(n,\mathbb{K}))$ *and* $X = X^i \frac{\partial}{\partial u^i} \in T_{\mathbf{0}}(\mathrm{T}_n)$ *and the above actions* L *are respectively*

$$\xi_X = X^i_j v^j \frac{\partial}{\partial v^i} \in \mathfrak{X}(V) \subseteq \mathfrak{X}(M) \tag{6.23a}$$
$$\xi_X = X^i \frac{\partial}{\partial v^j} \in \mathfrak{X}(V) \subseteq \mathfrak{X}(M). \tag{6.23b}$$

Proof. Since the mapping $a \mapsto a \rtimes \mathbf{0}$ (resp. $t \mapsto 1\!\!1 \rtimes t$) realizes a homeomorphism from $\mathrm{GL}(n,\mathbb{K})$ (resp. T_n) on $\mathrm{GA}(n,\mathbb{K})$, we can assert that instead of (6.21) now we have the equations

$$(T_a(L^x))\Big(Y^i_j \frac{\partial}{\partial y^i_j}\Big|_a\Big) = Y^i_j v^j(x)\frac{\partial}{\partial v^i}\Big|_{L(a,x)} \in T_{L(a,x)}(M) \tag{6.24a}$$
$$(T_t(L^x))\Big(Z^i \frac{\partial}{\partial u^i}\Big|_a\Big) = Z^i v^j(x)\frac{\partial}{\partial v^i}\Big|_{L(t,x)} \in T_{L(t,x)}(M) \tag{6.24b}$$

for respectively the left actions (6.22a) and (6.22b). The equations (6.23) follow immediately from here and definition 6.1. □

Now we shall record the evident analogue of corollary 6.4.

Corollary 6.5. *A vector field on C^1 manifold M is a linear (resp. constant) vector field relative to a chart (V, v) of M if and only if it reduces on V to a fundamental vector field of the general linear (resp. translation) group* $\mathrm{GL}(n, \mathbb{K})$ *(resp.* T_n*),* $n = \dim M$*, represented on* V *via the left action* (6.22a) *(resp.* (6.22b)*).*

At last, we notice that the results of subsections 6.1–6.3 are special cases of the above ones when $M = \mathbb{R}^n$ and $(V, v) = (\mathbb{R}^m, u)$, with $\{u^i\}$ being the standard Cartesian coordinate system on $\mathbb{R}^n$.

6.5. Examples of "non-standard" actions of T_n and $\mathrm{GL}(n, \mathbb{R})$

The fundamental vector fields generally depend on the concrete action of the group considered as it is clear from definition 6.1. The purpose of the following lines is to illustrate this fact as well as an exception of it.

Consider the (left/right) action

$$\lambda\colon \mathrm{T}_n \times \mathbb{R}^n \to \mathbb{R}^n\colon (t, x) \mapsto \lambda(t, x) = x\,\mathrm{e}^{s\cdot t} \tag{6.25}$$

of the translation group T_n on $\mathbb{R}^n$. Here $s = (s_1, \dots, s_n)^\top \in \mathbb{R}^n$ is a fixed element and $s\cdot t := \sum_i s_i t^i = s_i t^i$ is the Euclidean product of s and t. Using the notation of subsection 6.2, one finds that the fundamental vector field corresponding to $X = x^i \frac{\partial}{\partial z^i} \in T_{\mathbf{0}}(\mathrm{T}_n)$ and λ is

$$\xi_X = (X^i s_i) x^k \frac{\partial}{\partial u^k} \in \mathfrak{X}(\mathbb{R}^n). \tag{6.26}$$

Therefore linear vector fields of the type $cx^k \frac{\partial}{\partial u^k}$, c being a real constant, are fundamental vector fields of T_n and λ for suitable choice of X and/or s and *vice versa*. Therefore the fundamental vector fields of T_n relative to the representation (6.25) are linear vector fields while the ones relative to representation via translations are constant vector fields.

As a second example, let us investigate the (left) action

$$\lambda\colon \mathrm{GL}(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n\colon (a, x) \mapsto \lambda(t, x) = ax(\det a)^q, \tag{6.27}$$

for some number $q \in \mathbb{N} \cup \{0\}$, of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$; the case $q = 0$ being the one considered in subsection 6.1. Calculating the appearing in (6.2) derivatives (notice that $\frac{\partial \det a}{\partial a_i^j}\big|_{a=\mathbb{1}} = \delta_j^i$ with δ_j^i being the Kronecker deltas), we see that the fundamental vector field corresponding to $\mathrm{GL}(n, \mathbb{R})$, the action (6.27) and a vector $X \in T_{\mathbb{1}}(\mathrm{GL}(n, \mathbb{R}))$ is

$$\xi_X = (X_j^i x^j + q X_j^j x^i) \frac{\partial}{\partial u^i} = (X_j^i + q X_k^k \delta_j^i) x^j \frac{\partial}{\partial u^i}. \tag{6.28}$$

The set of these fundamental vector fields coincides with the one of linear vector fields as, if $C_j^i = X_j^i + q X_k^k \delta_j^i$, then $X_j^i = C_j^i - \frac{q}{1+qn} C_k^k \delta_j^i$. Hence it coincides with the one of $\mathrm{GL}(n, \mathbb{R})$ represented on $\mathbb{R}^n$ via left multiplication. However, the particular fundamental vector fields corresponding to a concrete vector $X \in T_{\mathbb{1}}(\mathrm{GL}(n, \mathbb{R}))$ are different for the two actions considered (unless $q = 0$ when they are identical).

7. Conclusion

As we already said in section 1, this paper reviews the concepts of affine and fundamental vector fields on a manifold. The main conclusions from it are that the affine, linear and constant vector fields on a manifold are in a bijective correspondence with the fundamental vector fields on it of respectively general affine, general linear and translation groups (locally)

represented on the manifold via the described in this work left actions; in a case of the manifold $\mathbb{K}^n = \mathbb{R}^n, \mathbb{C}^n$, the actions mentioned have the usual meaning of affine, linear and translation transformations.

Equations (6.2) can serve as a ground for studding a problem inverse to the one of finding fundamental vector fields, viz. to be found a Lie group and its action on a manifold if some set of vector fields plays a role of set of its fundamental vector fields. Precisely, given numbers $X^\mu \in \mathbb{K}$, $\mu = 1, \dots, N \in \mathbb{N}$, and vector fields [9]

$$\xi_X = X^\mu f^k_\mu \frac{\partial}{\partial u^k} \tag{7.1}$$

for some functions f^k_μ on M. Does there exists a Lie group G with $\dim G = N$ and a left/right action of G on M for which ξ_X is the fundamental vector field corresponding to $X = X^\mu \frac{\partial}{\partial y^m} \in T_e G$? In particular, the group G can be given and one should look for the existence/non-existence of (one or more) actions with the last property.

Comparing (7.1) and, e.g., (6.2a), we get

$$u^k \circ \lambda(a,x) = u^k(x) + f^k_\mu(x)[y^\mu(a) - y^\mu(e)] + f^k_{\mu\nu}(a,x)[y^\mu(a) - y^\mu(e)][y^\nu(a) - y^\nu(e)] \tag{7.2}$$

due to $\lambda(e,x) \equiv x$. Here $f^k_{\mu\nu}$ and their first partial derivatives relative to y^μ are bounded functions. To define a left action λ via this equation one needs to ensure that $\lambda_{ab} = \lambda_a \circ \lambda_b$ for all $a, b \in G$. As a result of (7.2), this requirement is equivalent to

$$\begin{aligned} f^k_\mu(x)[y^\mu(ab) - y^\mu(e)] + f^k_{\mu\nu}(ab,x)[y^\mu(ab) - y^\mu(e)][y^\nu(ab) - y^\nu(e)] \\ = f^k_\mu(x)[y^\mu(b) - y^\mu(e)] + f^k_{\mu\nu}(b,x)[y^\mu(b) - y^\mu(e)][y^\nu(b) - y^\nu(e)] \\ + f^k_\mu(\lambda(b,x))[y^\mu(a) - y^\mu(e)] + f^k_{\mu\nu}(a,\lambda(b,x))[y^\mu(a) - y^\mu(e)][y^\nu(a) - y^\nu(e)] \end{aligned} \tag{7.3}$$

which should hold for all $a, b \in G$ and $x \in M$. This is a system of equations for the left action λ (involved directly and via its expansion (7.2)) and the for the multiplication $G \times G \ni (a,b) \mapsto ab \in G$ in G.

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